Numerical General Relativistic Hydrodynamics

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We review the methods used to numerically solve fluid equations in general relativity. After discussing physical interpretations, we qualitatively analyze results from a simulation performed using these methods.

I. MOTIVATION

Einstein's general theory of relativity could easily be mistaken for a beautiful way of expressing Newton's laws of gravity. What causes Einstein's theory to depart from this picture is it makes measurable predictions, e.g. the precise rate of precession of Mercury's orbit. To take a pessimistic view, this fact is actually an unfortunate one for astrophysicists, because the mathematical form of Einstein's theory is much more cumbersome than that of Newton. In other words, we are forced to solve more difficult equations when modeling real physical phenomena in the regime of strong gravitational fields. Important applications include black hole accretion, massive collisions, and almost any problem in cosmology. The focus of this work will be on the first of these applications, that of matter modeled as a perfect fluid, accreting onto a black hole.

II. TIME EVOLUTION OF THE ENERGY-MOMENTUM TENSOR

We do not attempt to solve Einstein's field equations. Our goal will be to take a known solution these equations, and find the trajectory of a test fluid (low density in comparison to the black hole mass). For the problem of matter accretion, this is often a good approximation. The part of Einstein's equation that we do use is the fact that the energy-momentum tensor is set equal to a manifestly divergenceless quantity, the einstein tensor. In other words, the local conservation law

$$\nabla_\mu T^{\mu}_{\cdot\,\nu} = 0 \qquad (1)$$

can be viewed as four components of the Einstein equation[1]. The energy-momentum tensor will be that of a perfect fluid:

$$T^{\mu}_{\cdot\,\nu} = \rho h u^\mu u_\nu + P \delta^\mu_\nu \qquad (2)$$

where P is the pressure, ρ is the density, u is the four-velocity as a function of position and time, and h is the specific enthalpy of the fluid:

$$h = 1 + \epsilon + P/\rho \qquad (3)$$

and we are using the equation of state of a perfect fluid:

$$P = (\gamma - 1)\rho\epsilon \qquad (4)$$

We will also be using conservation of mass:

$$\nabla_\mu(\rho u^\mu) = 0 \qquad (5)$$

Solving (1) and (5) can be done using a flux-conservative method, but they have to first be expressed in the proper form. Fortunately, this is not very difficult. (1) can be written in the following way[2]:

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{(-g)}\,T^{\mu}_{\cdot\,\nu})-\Gamma^{\alpha}_{\mu\nu}T^{\mu}_{\cdot\,\alpha}=0 \tag{6}$$

This makes it easy to separate the temporal and spatial derivatives. Assuming the metric does not change with time (this is part of our assumption that we are modeling a test fluid near a black hole):

$$\partial_0(T^{0}_{\cdot\,\nu})+\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}\,T^{i}_{\cdot\,\nu})=\Gamma^{\alpha}_{\mu\nu}T^{\mu}_{\cdot\,\alpha} \tag{7}$$

This is the flux-conservative form of (1). (5) can similarly be manipulated:

$$\partial_0(\rho u^0)+\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}\,\rho u^i)=0 \tag{8}$$

Since ν runs over four dimensions, (7) and (8) give us five conserved quantities, with equations in the following form:

$$\partial_0 U+\frac{1}{\sqrt{-g}}\partial_i(\sqrt{-g}\,F^i)=S \tag{9}$$

Equations of the form (9) are naturally treated by Riemann solvers[3] (the only additional complication here is the geometry of our finite volume element, which manifests itself as the √-g in (9)). In our case, we have the following formulas for the conserved quantities, fluxes and source terms (For the purposes of studying this equation, we assume a diagonal metric. This is not a good general assumption, as the most important matter accretion problems are for a Kerr black hole, which does not admit a diagonal metric, but for the purposes of understanding each term in (9), we do not need to add complications like this):

$$\begin{array}{cccc} & \underline{\quad U\quad} & \underline{\quad F^i\quad} & \underline{\quad S\quad}\\ D & W\rho & W\rho v_i & 0\\ S_j & W^2\rho h v_j g_{jj} & W^2\rho h v_i v_j g_{jj}+P & \Gamma^{\alpha}_{\mu j}T^{\mu}_{\cdot\,\alpha}\\ E & W^2\rho h g_{00}+P & W^2\rho h g_{00}v_i & \Gamma^{\alpha}_{\mu 0}T^{\mu}_{\cdot\,\alpha}\end{array} \tag{10}$$

Here, D corresponds to relativistic density, S_j is the j[th] component of momentum density, and E corresponds to the energy density. We are using W to signify the lorenz factor (because γ is already being used to signify the adiabatic index). Since j runs from one to three, these are five conserved quantities. Actually, we don't generally work with E, but with τ = D – E, which is interpreted as the total energy minus mass energy. The final row then becomes:

$$\begin{array}{cccc}\tau & -W^2\rho h g_{00}-P-D & -W^2\rho h g_{00}v_i-D v_i & -\Gamma^{\alpha}_{\mu 0}T^{\mu}_{\cdot\,\alpha}\end{array} \tag{11}$$

II. THE NEWTONIAN-GALILEAN LIMIT

To get a better idea of the meaning of each term in (10), we study the weak-field, low velocity limit. Our result should reduce to Euler's equations in Newtonian gravity. The weak-field metric is[4]

$$ds^2=-(1+2\Phi)dt^2+(1-2\Phi)(dx^2+dy^2+dz^2) \tag{11}$$

where Φ is the gravitational potential. This is assumed to be a small perturbation from Minkowski space, so that |Φ|<<1. Additionally, the pressure should be small in comparison to the mass density. The Christoffel symbols are easily calculated, since the metric is diagonal:

$$\Gamma^{x}_{xx}=-\partial_x\Phi \quad \Gamma^{x}_{yy}=\partial_x\Phi \quad \Gamma^{x}_{tt}=\partial_x\Phi \quad \Gamma^{y}_{xy}=-\partial_x\Phi \quad \Gamma^{t}_{xt}=\partial_x\Phi \tag{12}$$

Now, to lowest order in v, |Φ|, and P, (10) reduces to:

$$\begin{array}{cccc} & \underline{\quad U\quad} & \underline{\quad F^i\quad} & \underline{\quad S\quad}\\ D & \rho & \rho v_i & 0\\ S_j & \rho h v_j & \rho h v_j v_i+P & -\rho\partial_j\Phi\\ \tau & (1/2)\rho v^2+\rho\Phi & \rho\Phi v_i+(1/2)\rho v^2 v_i & 0\end{array} \tag{13}$$

We see that a source term remains in the momentum equation, which is appropriate because momentum is not conserved in a gravitational field. Energy, however, is conserved, reflected in the cancellation of the source term for τ. This is because we are taking into account the gravitational potential energy in the expression for τ.

III. THE SWARZCHILD SOLUTION

Now that we've reproduced some expected results, let us turn our attention to a more interesting geometry: the Swarzchild metric. This metric will give us many more terms, partly because we are no longer ignoring higher-order corrections, and partly because this metric is naturally expressed in spherical coordinates. We shall assume an axial symmetry because this gives us one less flux to account, and also because we have not written a full three-dimensional solver, which requires much faster computing. The Conserved quantities and their respective flux functions are the following:

	U	F^r	F^θ	
D	$W\rho$	$W\rho v_r$	$W\rho v_\theta$	
S_r	$\dfrac{W^2\rho h v_r}{(1-\frac{2M}{r})}$	$\dfrac{W^2\rho h v_r^2}{(1-\frac{2M}{r})}+P$	$\dfrac{W^2\rho h v_r v_\theta}{(1-2M/r)}$	(14)
S_θ	$W^2\rho h v_\theta r^2$	$W^2\rho h v_\theta v_r r^2$	$W^2\rho h v_\theta^2 r^2+P$	
S_ϕ	$W^2\rho h v_\phi r^2\sin^2\theta$	$W^2\rho h v_\phi v_r r^2\sin^2\theta$	$W^2\rho h v_\phi v_\theta r^2\sin^2\theta$	
τ	$W^2\rho h(1-\frac{2M}{r})-P-D$	$W^2\rho h(1-\frac{2M}{r})v_r-D v_r$	$W^2\rho h(1-\frac{2M}{r})v_\theta-D v_\theta$	

And these are the source terms:

	U	S	
	D	0	
	S_r	$W^2\rho h(\frac{-M}{r^2}(1-\frac{2M}{r}+\frac{v_r^2}{(1-2M/r)})+r v_\theta^2+r v_\phi^2\sin^2\theta)+2P/r$	(15)
	S_θ	$(W^2\rho h v_\phi^2 r^2\sin^2\theta+P)\cot\theta$	
	S_ϕ	0	
	τ	0	

We are deviating from the conventions in the general literature; these calculations are usually carried out in terms of a lapse function and shift vector. This can be a handy way of expressing things, as it helps to solve the ambiguity in how to split space and time, and it makes it easier to go from Swarzchild to Kerr. However, it is useful to have alternate derivations, so that the reader is not forced to conform to every existing convention, merely because that is the only form in which all the equations in the literature are expressed. We should also note that "velocity" here means coordinate rate of change per unit time. For example, v_ϕ is actually the angular velocity about the z axis, not literally the axial component of velocity.

It will be informative to take a moment to understand the physical meaning of each source term. We shall find that each source term represents some kind of "fictitious force", because (1) tells us that all momenta are locally conserved. Thus, the source terms are really artifacts of our coordinate system, but of course they will still generally be important.

The first part of the radial source term,

$$W^2\rho h\left(-\frac{M}{r^2}\left(1-\frac{2M}{r}+\frac{v_r^2}{(1-2M/r)}\right)\right) \tag{16}$$

is the only source term which has a surviving remnant in the Newtonian-Galilean limit (13). This corresponds to the gravitational force, which in general relativity is interpreted as a fictitious force. The next term in the radial component,

$$W^2\rho h(r v_\theta^2+r v_\phi^2\sin^2\theta) \tag{17}$$

is a centrifugal force. This, too, is clearly fictitious. Turning our attention to the first part of the axial source term:

$$W^2 \rho h v_\phi^2 r^2 \sin\theta \cos\theta \tag{18}$$

This we recognize as the Coriolis force. Now, there are only two remaining sources to interpret. These are the source terms proportional to the pressure:

$$\begin{aligned} S_r^P &= 2\mathrm{P}/r \\ S_\theta^P &= P \cot\theta \end{aligned} \tag{19}$$

These terms compensate for the non-euclidean geometry of the finite volume cells in our lattice. Because the opposite faces of each cell have different surface areas, there will be a different force calculated on each face, even if the pressure is constant. We know in reality this is because the other four faces are not perpendicular to the two opposite faces in question, and so they can contribute pressures along this dimension. These pressures are exactly what these source terms represent, and these, too, can be thought of as fictitious (net) forces.

IV. NUMERICAL RESULTS

We look at a 2+1 dimensional general relativistic hydrodynamics simulation employing these methods. This was written in C, and runs in parallel by breaking up the solution space into concentric spheres and passing boundary data between processes. When we say "2+1 dimensional", we really mean "3+1 dimensional", but we are imposing axial symmetry, hence no axial flux term in (14).

The simulation models accretion onto a Swarzchild black hole. We start by looking at rather simple initial conditions (no angular momentum), so that we can test that the procedure works. Upon analyzing the steady-state solution, we see that the matter accretion rate

$$\dot{M} = r^2 W \rho v_r \tag{20}$$

is constant throughout space by the time things settle down. This is a basic physical benchmark, which really just tells us that our flux-conservative method is working properly (this fact is guaranteed by our formulas above; just substitute the first line of (14) into (9), and look at the steady-state solution, and (20) will be seen to be constant in space). In other words, this is not a test for convergence, but a test for bugs. Fortunately, we pass this test, and can move on to more substantial challenges.

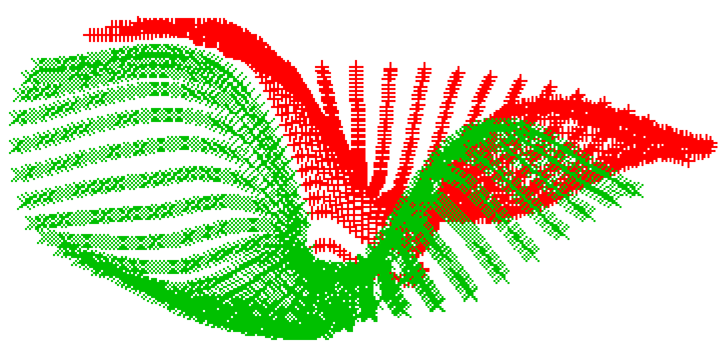

Fig. 1 Radial component of velocity during oscillation.

We very briefly (and very tentatively) explore the question of oscillations in spherical accretion. Accreting matter is known to oscillate[5] (and this simulation also finds this to be the case), at frequencies roughly around the mHz to Hz range. We find frequencies of the order 10~100 Hz for spherically symmetric oscillation about a black hole of 10 solar masses. This result is qualitative, and certainly not exact (real accretion is, of course, not spherically symmetric, and involves many other factors), but it is nice to reproduce oscillation in vaguely the same frequency ballpark.

We finally test our code with some nontrivial initial conditions and look at the results. There are two interesting periods of time: the initial instability caused by the initial conditions, and the convergence to steady-state. We note that the black hole immediately rejects the angular velocity given to the fluid; the angular momentum is dissipated along the z axis. The axial angular velocity quickly goes to zero, and we are left with only radial velocity. This is not entirely surprising, because our black hole is not rotating, and the inflowing matter has purely radial velocity. When the code is made to be more stable, it will be worthwhile seeing how the black hole reacts to inflow conditions with incoming angular momentum.

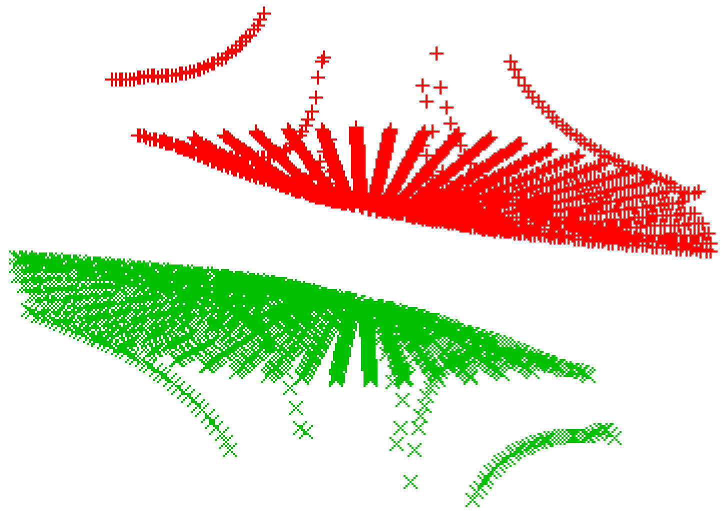

Fig. 2 Axial angular velocity being quickly ejected along the z-axis (at early times).

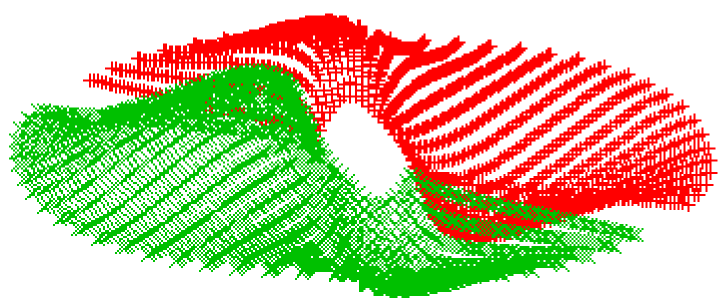

Fig. 3 Azimuthal angular velocity oscillating at early times.

The steady-state regime is the one that agrees with (20). After ejecting angular momentum, the accretion begins a gradual relaxation process, eventually finding a steady state solution. This is a simplification, because we assume that the incoming matter flows in symmetrically and at a constant rate. For realistic accretion, there will not be a constant inflow, and hence no true steady state. This is one likely reason that real accretion oscillations can persist indefinitely, while they are only found at early times in this simulation.

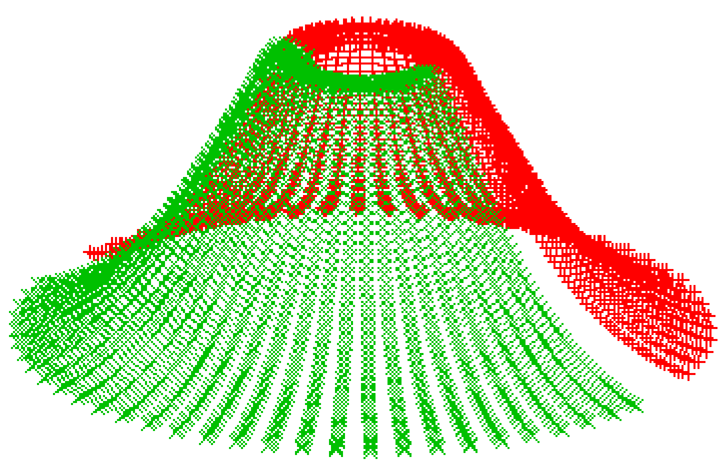

Fig. 4 The steady-state density distribution (late times).

V. CONCLUDING REMARKS

We numerically addressed the problem of a test fluid orbiting a Swarzchild black hole with nontrivial initial conditions, using flux-conservative methods in 2+1 dimensional general relativistic hydrodynamics. We now take a moment to remark on a few directions that we could potentially take this approach. It would have been interesting to try using perturbation theory to get weak self-gravity without solving Einstein's equations. Alternatively, if we had decided to “bite the bullet” and solve Einstein, it might have been interesting to try getting more out of this by including a symmetric fifth dimension and using Kaluza-Klein theory to give us magnetic fields for free. As for less lofty goals, another interesting approach would be to use multiple coordinate patches to cover the space-time manifold (since we are already running things in parallel, this would not cost us much in overhead). This would save us from having to surgically remove coordinate singularities which might have interesting physics (for example, the angular momentum was transported out of our black hole along the z axis, which was artificially removed because of the coordinate singularity in spherical coordinates. It is generally known that a great deal of interesting physics goes on along this axis).

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