Kaluza's Unifying Theory

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We look at the ideas proposed by Kaluza and Klein, specifically how electromagnetism can be derived from a five-dimensional spacetime manifold, after imposing certain restrictions. As an example, we derive the Reissner-Nordström metric for an electrically charged black hole. We also look at some of the physical interpretation involved in assuming a fifth dimension with a translational symmetry.

I. AN INTRODUCTION ON THE INTERPLAY BETWEEN GRAVITY AND ELECTROMAGNETISM

Einstein's General Relativity has no difficulty meshing with Classical Electromagnetism. The ease of this is simply due to the equivalence principle: to add gravity to any theory, simply write the laws in tensorial form, then jump out a window. While freely falling, you're locally in an inertial frame, so gravity has no immediate effect on things, and the laws you've written down should still hold. Once this has been established, make a coordinate transformation to a non-inertial frame, and you suddenly find your theory coupled to gravity. Specifically, Maxwell's equations get the relativistic form:

$$F^{\nu\mu}_{\cdot\cdot;\mu} = J^{\nu} \qquad F_{[\nu\lambda;\mu]} = 0 \qquad F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta} \tag{1}$$

This gives Einstein's effect on Maxwell. We still have to show Maxwell's effect on Einstein, which manifests itself as a term in the stress energy tensor, which is the source for spacetime curvature:

$$T^{\mu\nu}_{EM} = F^{\mu\lambda} F^{\nu}_{\cdot\lambda} - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \qquad R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \kappa T^{\mu\nu} \tag{2}$$

This is generally a coupled set of nonlinear, second-order differential equations. Practically speaking, this is all we need to know, and we can start calculating things. Theodor Kaluza, however, found a way to look at this interplay between forces in a completely different way.

II. KALUZA'S THEORY

The inspiration that Kaluza[1] had was to extrapolate the laws of general relativity into a fifth dimension. The five-dimensional metric takes the following form:

$$\begin{aligned} ds^2 &= \hat{g}_{\hat{\mu}\hat{\nu}} dx^{\hat{\mu}} dx^{\hat{\nu}} \\ &= g_{\mu\nu} dx^{\mu} dx^{\nu} + (dy + \kappa A_{\mu} dx^{\mu})^2 \end{aligned} \tag{3}$$

We are following the general literature by using carets over quantities in the total five-dimensional space, and indices which run over all five dimensions. For quantities which are interpreted as properties of the projection to 4-d spacetime and indices that only run through these four dimensions, We have removed the carets. Clearly above, "y" denotes this 5^{th} dimension, but we would need to use more general notation if we were treating more than 5 dimensions. This is entirely possible, and necessary for generalization to forces other than electromagnetism, but this will not be attempted here (Toms [2] does this). Also please note that we are using the metric signature (- + + + +), so that "y" is interpreted as a spatial dimension.

Kaluza made two assumptions on this metric; that $g_{yy} = 1$, and that all other components of the metric are independent of y. Herein lies the greatest detractor to kaluza's theory: built into this metric is the condition that we cannot detect this fifth dimension with any experiment. This is a problem for two reasons: one is that this condition seems fairly artificial, on the face of it. Secondly, and probably more importantly, the theory appears guaranteed to do nothing better than reproduce equations (1) and (2). There is no Kaluza equivalent of the experiments Einstein suggested to test General Relativity, e.g. the deflection of light by the sun, or the precession of Mercury's orbit. As we will discuss in section four, Oskar Klein[3] was eventually able to address the first concern, that this symmetry might seem artificially imposed, but the latter problem persists (with an important exception, to be discussed later). Regardless of the theoretical problems it creates, enforcing this symmetry immediately leads to some interesting results. For example, imagine we make the most general coordinate transformation allowed by the rules that Kaluza has set forth. Arbitrary coordinate transformations (independent of y) will be allowed in the non-Kaluza dimensions, but for this additional coordinate, the restricted class of transformations take the following form:

$$y \rightarrow y' = y + \Lambda(x^\mu) \tag{4}$$

If we insert this coordinate transformation in to the metric (3), we immediately get the result:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu + (dy + \frac{\partial \Lambda}{\partial x^\mu} dx^\mu + \kappa A_\mu dx^\mu)^2 \tag{5}$$

This transformation will not affect the form of the line element, if we also transform our definition of A_μ:

$$A_\mu \rightarrow A_\mu' = A_\mu - \frac{1}{\kappa} \frac{\partial \Lambda}{\partial x^\mu} \tag{6}$$

This we immediately identify with an electromagnetic gauge transformation, and conversely such gauge transformations are always identified with a transformation of the kaluza coordinate. Already, before any actual electromagnetism is exhibited, the theory shows great promise, in that it relates the idea of gauge invariance directly to the idea of coordinate covariance.

It is not too hard to show[2] that the metric (3) induces the following Ricci scalar:

$$\hat{R} = R - \frac{1}{4} \kappa^2 F^{\mu\nu} F_{\mu\nu} \tag{7}$$

where F is the field strength derived from A,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{8}$$

therefore the Einstein-Hilbert action for this metric is equal to that of the four-dimensional space, plus a contribution proportional to the Maxwell action from electromagnetism, and this gives us a numerical value for kappa:

$$\kappa = \pm 4\sqrt{\pi G} \tag{9}$$

While action integrals are often a very succinct way to get results, they can often cause one to arrive at a solution without realizing how he or she got there. So, in the interest of education, We'll accept this result as correct, but spend a little more time looking at how all of this actually affects Einstein's equation. In the four space-time coordinates, we see a vacuum, since we are no longer thinking of electromagnetism as contributing a source term:

$$\hat{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \hat{R} = 0 \tag{10}$$

Actually, in vacuum, this equation also holds for the off-diagonal curvature terms between spacetime and the kaluza dimension:

$$\hat{R}^{\mu y} - \frac{1}{2} \hat{g}^{\mu y} \hat{R} = 0 \tag{11}$$

However, we do not enforce any condition for the y-y component of the einstein tensor. If we did, it would actually require zero electromagnetic field, as we'll show in an example in the next section. It is worth understanding physically why we allow this freedom for this component of the Einstein tensor. This can be understood from the point of view of the symmetry requirement on the Kaluza dimension. Since we have this perfect symmetry, the flux through this dimension should not be a directly measurable quantity.

The y-y component of the Einstein tensor keeps our solution from reducing back to the Einstein vacuum equation. It is in this component that the information about the electromagnetic energy density is hiding. In other words, we've really just moved the source term in (2) to the left-hand side and incorporated it into our curvature tensor. This can be seen by taking the trace of the Einstein equation:

$$\hat{R}^{\hat{\mu}}_{\cdot\,\hat{\mu}} - \frac{1}{2}\hat{g}^{\hat{\mu}}_{\cdot\,\hat{\mu}}\hat{R} = \hat{R}^{y}_{\cdot\,y} - \frac{1}{2}\hat{R} = \hat{R} - \frac{5}{2}\hat{R} \qquad (12)$$
$$\hat{R}^{y}_{\cdot\,y} = -\hat{R}$$

So, Einstein's equation now looks like the following in the lower four coordinates:

$$\hat{R}^{\mu\nu} + \frac{1}{2}\hat{g}^{\mu\nu}\hat{R}^{y}_{\cdot\,y} = 0 \qquad (13)$$

or, to repackage things:

$$\hat{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\tilde{R} = T^{\mu\nu}_{eff} \qquad (14)$$
$$\tilde{R} = R^{\mu}_{\cdot\,\mu}, \qquad T^{\mu\nu}_{eff} = \frac{1}{2}\hat{g}^{\mu\nu}(\hat{R})$$

Where the R with a tilde is a trace of the full 5-D Ricci tensor over only four of its dimensions (not to be confused with an uncareted R, which would be the trace of a ricci tensor which was *calculated* using only the lower four dimensions). We have thus expressed Einstein's equation in terms of an effective energy-momentum tensor. This way of writing things allows us to interpret the y-y component of the einstein tensor as a pressure, which we might think of as an internal energy density. This is a nice way of interpreting things, because we have an additional dimension where this internal energy can hide; as we noted before, the flux along the Kaluza dimension is not directly measurable (Of course, the electromagnetic field is measurable, but we are thinking of that as an indirect measurement of flux along the Kaluza dimension).

III. THE SPHERICALLY SYMMETRIC CASE

It will be worth working through an example to see how all of these ideas play out. Our goal will be to find a spherically symmetric stationary solution to Einstein's equations using a five-dimensional Kaluza metric. Rather than go through a rigorous proof like that of Birkoff[4], We begin by assuming the following metric:

$$ds^2 = (dy + \gamma(r)\,dt)^2 - e^{2\alpha(r)}dt^2 + e^{2\beta(r)}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)\,d\phi^2 \qquad (15)$$

This is similar to the starting point Carroll[5] uses in deriving the Swarzchild solution. The first thing to notice about this metric is that we have omitted the cross-term involving dr dy. This term can be moved easily via a gauge transformation (6). It is also worth noting the other cross terms we've omitted, namely dy dθ and dy dϕ. These terms would be allowed if we did not want parity symmetry. If we were to loosen this restriction, it would lead to a solution with a magnetic monopole in general, which is certainly a worthwhile thing to study, but this is not the calculation that we shall be doing today. Now notice that while this metric has rotational symmetry, time translational symmetry, parity symmetry, and of course the symmetry along the kaluza dimension, it does not have time reversal symmetry. In order to get a time reversal symmetry, we'd have to combine it with a kaluza-reversal transformation y -> -y, which is tantamount to charge conjugation. If we allowed cross-terms involving a combination of spatial components with the kaluza dimension, we'd see that the symmetry was actually CPT; charge conjugation combined with parity and time-reversal. Note we are able to make statements like this before saying anything about the nature of the current density J of our theory.

Also worth noting is that in writing down this line element, I've avoided the language of electromagnetism, because we want the laws to fall out of this formalism naturally. Hence, what we called κA_0 before, we now call gamma. The details of the calculation of the Riemann curvature tensor can be found in the included appendix. The resulting Ricci tensor looks like the following (in normalized coordinates, see the appendix for details):

$$
\begin{aligned}
\hat{R}_{00} &= (\alpha''+(\alpha')^2-\alpha'\beta')e^{-2\beta}+\frac{2\alpha'}{r}e^{-2\beta} \\
\hat{R}_{11} &= -(\alpha''+(\alpha')^2-\alpha'\beta')e^{-2\beta}+\frac{2\beta'}{r}e^{-2\beta} \\
\hat{R}_{22} &= -\frac{\alpha'}{r}e^{-2\beta}+\frac{\beta'}{r}e^{-2\beta}+\frac{1}{r^2}(1-e^{-2\beta}) \\
\hat{R}_{33} &= -\frac{\alpha'}{r}e^{-2\beta}+\frac{\beta'}{r}e^{-2\beta}+\frac{1}{r^2}(1-e^{-2\beta}) \\
\hat{R}_{50} &= -\frac{1}{2}e^{-\alpha-2\beta}(\gamma''-\gamma'\beta'-\gamma'\alpha'+2\gamma'/r) \\
\hat{R}_{55} &= -\frac{1}{2}\gamma'^2 e^{-2\alpha-2\beta}
\end{aligned}
\tag{16}
$$

Now we express the Einstein equation using (10-12):

$$\hat{R}=-\hat{R}^{5}_{.5}=\frac{1}{2}\gamma'^2 e^{-2\alpha-2\beta} \tag{17}$$

$$
\begin{aligned}
(\alpha''+(\alpha')^2-\alpha'\beta')e^{-2\beta}+\frac{2\alpha'}{r}e^{-2\beta}+\frac{1}{4}\gamma'^2 e^{-2\alpha-2\beta} &= 0 \quad (A) \\
-(\alpha''+(\alpha')^2-\alpha'\beta')e^{-2\beta}+\frac{2\beta'}{r}e^{-2\beta}-\frac{1}{4}\gamma'^2 e^{-2\alpha-2\beta} &= 0 \quad (B) \\
-\frac{\alpha'}{r}e^{-2\beta}+\frac{\beta'}{r}e^{-2\beta}+\frac{1}{r^2}(1-e^{-2\beta})-\frac{1}{4}\gamma'^2 e^{-2\alpha-2\beta} &= 0 \quad (C) \\
-\frac{\alpha'}{r}e^{-2\beta}+\frac{\beta'}{r}e^{-2\beta}+\frac{1}{r^2}(1-e^{-2\beta})-\frac{1}{4}\gamma'^2 e^{-2\alpha-2\beta} &= 0 \quad (D) \\
-\frac{1}{2}e^{-\alpha-2\beta}(\gamma''-\gamma'\beta'-\gamma'\alpha'+2\gamma'/r) &= 0 \quad (E)
\end{aligned}
\tag{18}
$$

This looks foreboding at first, but it's not hard to solve. Adding (A) to (B) we get

$$\alpha'=-\beta' \tag{19}$$

which, up to a rescaling of the time coordinate, is the sam e thing as:

$$\alpha=-\beta \tag{20}$$

This makes (E) easy to solve. In fact, we see that (E) is just the radial component of Laplace's equation in spherical coordinates; here is where we solve Maxwell's equations (in general this is much harder, of course, but the static spherically symmetric solution is easy). Solving (E), we get a formula for gamma (after performing a kaluza gauge transformation to absorb an integration constant):

$$\gamma(r)=\kappa\frac{q}{4\pi r} \tag{21}$$

We have re-introduced kappa from (3) so as to interpret this as the electric potential of a point charge, which is exactly what we would expect to find. Finally, we can use either (C) or (D) to get a formula for beta:

$$\left(\frac{2\beta'}{r}-\frac{1}{r^2}\right)e^{-2\beta}+\frac{1}{r^2}-\frac{1}{4}\frac{\kappa^2 q^2}{16\pi^2 r^4}=0 \tag{22}$$

This is not hard to solve, once we write it in the following form:

$$\partial_r(r e^{-2\beta})=1-\frac{\kappa^2 q^2}{64\pi^2 r^2} \tag{23}$$

Hence:

$$e^{-2\beta}=1+\frac{const}{r}+\frac{\kappa^2 q^2}{64\pi^2 r^2} \tag{24}$$

This is almost in a recognizable form; for q = 0 we know that this must reduce to the Swarzchild solution, where this integration constant is equal to -2GM. Also, we can use (9) for the value of kappa:

$$e^{-2\beta}=1-\frac{2\mathrm{GM}}{r}+\frac{G q^2}{4\pi r^2}$$

And so we can write down the four-dimensional spacetime line element:

$$ds^2=-(1-\frac{2\mathrm{GM}}{r}+\frac{G q^2}{r^2})dt^2+\frac{dr^2}{(1-\frac{2\mathrm{GM}}{r}+\frac{G q^2}{r^2})}+r^2 d\theta^2+r^2\sin^2\theta\, d\phi^2 \qquad (25)$$

This is the well-established Reissner-Nordström metric[6] for an electrically charged black hole.

IV. INTERPRETATION AND EXTRAPOLATION

It is difficult to give an account of Kaluza's theory without at least mentioning the important work of Oskar Klein. Before we get to Klein's interpretation however, let us visit a point about predictions. At first glance, it seems that Kaluza theory only reproduces Einstein-Maxwell, and doesn't make any predictions of its own. There is at least one exception to this statement: Kaluza theory can predict charge quantization. It is known that Maxwell's electromagnetism predicts charge quantization, provided the existence of magnetic monopoles, along with some basic conditions on the quantum mechanical interpretation of gauge transformations. This is, at its most basic level, due to the topology of the sphere (which is a deformation retract of the space surrounding a magnetic charge). However, Kaluza theory does not need these assumptions to get charge quantization; it gets this result from the topology of the kaluza dimension (see [7]). This is a subtle distinction which separates it from Maxwell-Einstein, since Maxwell alone says nothing about the constituent nature of the current density.

When Kaluza first introduced his theory, this extra dimension was more of a mathematical construct than a real physical idea; since it was imposed that there is a symmetry along this direction, we didn't worry about the fact that we “really” live in a 3+1 dimensional spacetime. Oskar Klein brought an interpretation to this which was so powerful, Kaluza's theory is nowadays called “Kaluza-Klein” theory. Klein's idea was that we are allowed extra dimensions, but they are “curled up”, in other words periodic with a small periodicity:

$$y\sim y+L \qquad (26)$$

For a value of L smaller than any experimental detector, this periodic relation on y will look like a continuous translational symmetry. Klein, of course, gave a much more detailed argument than this, but even without going into the mathematics it is not hard to see that a small periodic fifth dimension leads effectively to Kaluza's conditions at macroscopic scales. Thus, spacetime in Kaluza-Klein theory has the (local) topology

$$M_{KK}\approx M_4\times S^1 \qquad (27)$$

This gives us electromagnetism. Can we perform a similar procedure for the remaining two forces? It turns out we can, but of course we have to know something about the gauge groups of these forces. In field theory, we associate the circle in (27) with electromagnetism's gauge group U(1), and it is not too hard to see that when we include all the other forces, the Kaluza-Klein spacetime manifold locally takes the form:

$$M_{KK}\approx M_4\times U(1)\times SU(2)\times SU(3) \qquad (28)$$

(in the above relation we are thinking of these lie groups as manifolds, so that U(1) is really just a circle). In an attempt to steer things away from a lengthy digression into the mathematics of lie algebras and fiber bundles, the statement (28) will be all that will be said here about forces other than electromagnetism.

V. CONCLUSION

5-dimensional Kaluza-Klein theory is equivalent to Einstein-Maxwell theory, in addition to being an extremely beautiful idea. It gives us profound insight on the nature of spacetime, and its relationship to the four forces of nature. The results it gives for electromagnetism are not very difficult to work through; we were able to find the Reissner-Nordstrom metric using a very different approach from the usual “textbook” method, i.e. we solved equations (10) and (11) rather than (1) and (2). This extra dimension and its imposed symmetry can be thought of as real physical properties of spacetime, provided we accept Klein's view, that the extra dimensions are periodic and small enough that the discrete symmetry (26) behaves like a continuous symmetry at larger scales.

APPENDIX:

We calculate the components of the Riemann curvature tensor from the line element (15). The first step is to write this in normalized coordinates:

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 + (\omega^5)^2 \tag{A1}$$

where the coordinates are defined by the following:

$$\begin{aligned}
\omega^0 &= e^{\alpha(r)}\, dt &&= -\omega_0 \\
\omega^1 &= e^{\beta(r)}\, dr \\
\omega^2 &= r\, d\theta \\
\omega^3 &= r \sin\theta\, d\phi \\
\omega^5 &= dy + \gamma(r)\, dt
\end{aligned} \tag{A2}$$

We then use the first Cartan structure equation with no torsion to find the connection one-forms:

$$d\omega_\mu + \omega_{\mu\nu} \wedge \omega^\nu = 0 \tag{A3}$$

$$\begin{aligned}
d\omega_0 &= -\alpha' e^{\alpha} \omega^1 \wedge \omega^0 \\
d\omega_1 &= 0 \\
d\omega_2 &= \frac{1}{r} e^{-\beta} \omega^1 \wedge \omega^2 \\
d\omega_3 &= \frac{1}{r} e^{-\beta} \omega^1 \wedge \omega^3 + \frac{1}{r} \cot\theta\, \omega^2 \wedge \omega^3 \\
d\omega_5 &= \gamma' e^{-\alpha-\beta} \omega^1 \wedge \omega^0
\end{aligned} \tag{A4}$$

The structure equation (A3) is satisfied by the following connection forms:

$$\begin{aligned}
\omega_{05} &= \frac{1}{2}\gamma' e^{-\alpha-\beta}\omega^1 &&= \frac{1}{2}\gamma' e^{-\alpha}\, dr \\
\omega_{15} &= -\frac{1}{2}\gamma' e^{-\alpha-\beta}\omega^0 &&= -\frac{1}{2}\gamma' e^{-\beta}\, dt \\
\omega_{01} &= -\alpha' e^{-\beta}\omega^0 + \frac{1}{2}\gamma' e^{-\alpha-\beta}\omega^5 &&= -\alpha' e^{\alpha-\beta}\, dt + \frac{1}{2}\gamma' e^{-\alpha-\beta}(dy + \gamma\, dt) \\
\omega_{21} &= \frac{1}{r} e^{-\beta}\omega^2 &&= e^{-\beta}\, d\theta \\
\omega_{31} &= \frac{1}{r} e^{-\beta}\omega^3 &&= e^{-\beta}\sin\theta\, d\phi \\
\omega_{32} &= \frac{1}{r}\cot\theta\, \omega^3 &&= \cos\theta\, d\phi
\end{aligned} \tag{A5}$$

We can now find the components of the curvature two-form from the second Cartan structure equation:

$$\Omega_{\mu\nu} = d\omega_{\mu\nu} + \omega_{\mu\cdot}^{\ \lambda} \wedge \omega_{\lambda\nu} \tag{A6}$$

We find the following components:

$$
\begin{aligned}
\Omega_{50} &= \frac{-1}{4}\gamma'^2 e^{-2\alpha-2\beta}\omega^0\wedge\omega^5 \\
\Omega_{51} &= \frac{1}{2}(\gamma''-\gamma'\beta'-\gamma'\alpha')e^{-\alpha-2\beta}\omega^1\wedge\omega^0+\frac{1}{4}e^{-2\alpha-2\beta}\omega^1\wedge\omega^5 \\
\Omega_{52} &= -\frac{1}{2r}\gamma' e^{-\alpha-2\beta}\omega^0\wedge\omega^2 \\
\Omega_{53} &= -\frac{1}{2r}\gamma' e^{-\alpha-2\beta}\omega^0\wedge\omega^3 \\
\Omega_{01} &= (-\alpha''-(\alpha')^2+\alpha'\beta'+\frac{1}{4}\gamma'^2 e^{-2\alpha})e^{-2\beta}\omega^1\wedge\omega^0+\frac{1}{2}(\gamma''-\gamma'\alpha'-\gamma'\beta')e^{-\alpha-2\beta}\omega^1\wedge\omega^5 \\
\Omega_{02} &= \frac{\alpha'}{r}e^{-2\beta}\omega^0\wedge\omega^2-\frac{1}{2r}\gamma' e^{-\alpha-2\beta}\omega^5\wedge\omega^2 \\
\Omega_{03} &= \frac{\alpha'}{r}e^{-2\beta}\omega^0\wedge\omega^3-\frac{1}{2r}\gamma' e^{-\alpha-2\beta}\omega^5\wedge\omega^3 \\
\Omega_{12} &= \frac{\beta'}{r}e^{-2\beta}\omega^1\wedge\omega^2 \\
\Omega_{13} &= \frac{\beta'}{r}e^{-2\beta}\omega^1\wedge\omega^3 \\
\Omega_{23} &= \frac{1}{r^2}(1-e^{-2\beta})\omega^2\wedge\omega^3
\end{aligned}
\tag{A7}
$$

Because we can relate the curvature two-form easily to the Reimann curvature,

$$\Omega_{\mu\nu}=\frac{1}{2}R_{\mu\nu\rho\sigma}\omega^\rho\wedge\omega^\sigma \tag{A8}$$

we arrive at the following (non-redundant) components for the Riemann tensor:

$$
\begin{aligned}
R_{5050} &= (1/4)\gamma'^2 e^{-2\alpha-2\beta} \\
R_{5151} &= -(1/4)\gamma'^2 e^{-2\alpha-2\beta} \\
R_{5101} &= -(1/2)(\gamma''-\gamma'\beta'-\gamma'\alpha')e^{-\alpha-2\beta} \\
R_{5202} &= -(1/2\mathrm{r})\gamma' e^{-\alpha-2\beta} \\
R_{5303} &= -(1/2\mathrm{r})\gamma' e^{-\alpha-2\beta} \\
R_{0101} &= (\alpha''+\alpha'^2-\alpha'\beta'-(1/4)\gamma'^2 e^{-2\alpha})e^{-2\beta} \\
R_{0151} &= -(1/2)(\gamma''-\gamma'\alpha'-\gamma'\beta')e^{-\alpha-2\beta} \\
R_{0202} &= (\alpha'/r)e^{-2\beta} \\
R_{0252} &= -(1/2\mathrm{r})\gamma' e^{-\alpha-2\beta} \\
R_{0303} &= (\alpha'/r)e^{-2\beta} \\
R_{0353} &= -(1/2\mathrm{r})\gamma' e^{-\alpha-2\beta} \\
R_{1212} &= (\beta'/r)e^{-2\beta} \\
R_{1313} &= (\beta'/r)e^{-2\beta} \\
R_{2323} &= (1-e^{-2\beta})/r^2
\end{aligned}
\tag{A9}
$$

The Ricci tensor is then easily found by contracting these indices, and this gives us the tensor components in (16).

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