Quantum Field Theory in Curved Spacetime

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1 The General Idea

The two great achievements of theoretical physics the past century, the general of relativity and the quantum theory of fields, are ideas of great depth and subtlety. These subtleties can be amplified in the juxtaposition encountered when attempting to apply both theories to a problem. While the relativistic quantization of the gravitational field is a mystery which still eludes us, we can already see some fascinating consequences when the gravitational field is treated classically, with quantum fields added to a gravitational background.

We take the simplest case of a scalar klein-gordon field. In flat spacetime, the lagrangian appears as

$$\mathscr{L} = -\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \tag{1}$$

This lagrangian can be upgraded to general relativity simply by the prescription that we write everything in a covariant way. However, nothing prevents us from adding additional terms involving the curvature:

$$\mathscr{L} = -\nabla_\mu \phi \nabla^\mu \phi - m^2 \phi^2 - \xi R \phi^2 \tag{2}$$

In fact, as we'll see a bit later, such a term can make a massless theory invariant under conformal transformations if we choose ξ appropriately. Of course, the first term hasn't really changed, since the covariant derivative of a scalar is just a partial derivative, but it's probably a good idea to write everything in a manifestly covariant way. The klein-gordon equation,

$$\partial_\mu \partial^\mu \phi - m^2 \phi = 0 \tag{3}$$

Becomes

$$\nabla_\mu \nabla^\mu \phi - m^2 \phi - \xi R \phi = 0 \tag{4}$$

or, in non-covariant form,

$$\partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) - \sqrt{g} m^2 \phi - \sqrt{g} \xi R \phi = 0 \tag{5}$$

Most of what we discuss here will be independent of details housed by the last two terms, so we set $m^2 = \xi = 0$ and the equation of motion is

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{6}$$

This equation of motion can in general be very ugly, and moreover the solutions can depend sensitively on the choice of coordinates. Instead of the plane wave states of the flat theory,

$$\phi \sim e^{-i\omega t + i\vec{k}\cdot\vec{x}} \tag{7}$$

We write our solutions as having a more generic form

$$\phi \sim g(\vec{k}) \tag{8}$$

where $\vec{k}$ is a label for the modes. Then our quantum field operator has the following form:

$$\phi = \int d^3k(a_k g(\vec{k}) + a_k^\dagger g^*(\vec{k})) \tag{9}$$

Where we have implicitly selected the positive frequency modes as coefficients for the annihilation operator. Now imagine we perform a coordinate transformation so that our metric looks different, and the equation of motion changes. Our quantum operator can then be expressed as:

$$\phi = \int d^3k(b_k h(\vec{k}) + b_k^\dagger h^*(\vec{k})) \tag{10}$$

where the $h(k)$ are the solutions to the equation in the new set of coordinates, and b is the annihilation operator in these coordinates. On the surface, there is nothing strange about any of this. We know, for example, we can write the solutions to the klein-gordon equation in spherical coordinates, and instead of plane waves we will get spherical harmonics. Our $h(k)$ can then be expressed as a linear combination of our $g(k)$'s, and therefore the creation operator in our new set of coordinates is just a linear combination of our old creaton operators. Something changes, however, when our coordinate choice includes a change in our time slicing. When we separate our positive frequency solutions from the negative frequency ones, we implicitly use our time slicing, and therefore this separation is coordinate-dependent. Since the $h(k)$ are a set of positive frequency modes specific to one coordinate system, we cannot say for sure that they can be expressed as a sum of positive-frequency modes in another system. In general, they can be expressed as a sum of positive and negative frequency modes:

$$h(\vec{k}) = \int d^3k'(\alpha_{kk'}g(\vec{k}') + \beta_{kk'}g^*(\vec{k}')) \tag{11}$$

and therefore the creation and annihilation operators are related by a Bogolubov transformation:

$$b_k = \int d^3k'(\alpha^*_{kk'}a_{k'} + \beta^*_{kk'}a^\dagger_{k'}) \tag{12}$$

This means that the vacuum state is not preserved under a general coordinate transformation. The a_k's don't annihilate the b vacuum, and the b_k's don't annihilate the a vacuum, and therefore the two vacuum states are in general distinct:

$$b(\vec{k})|0_A\rangle \neq 0 \tag{13}$$

$$|0_A\rangle \neq |0_B\rangle \tag{14}$$

So, a normal observer to coordinate system B who is sitting in the vacuum state of coordinate system A will detect particles. The number of particles he detects will be

$$\langle N_B(\vec{k})\rangle = \langle 0_A|b_k^\dagger b_k|0_A\rangle = \int d^3k|\beta_{kk'}|^2 \tag{15}$$

2 The Unruh Effect

This coordinate dependence of the vacuum state can be realized even in flat spacetime. The simplest example is the Unruh effect, that an accelerating observer in the vacuum state of Minkowski space will detect a thermal background. This can be thought of as a gravitational effect, since the observer experiences a force which he cannot distinguish from gravity (in other words, we don't need curvature to get gravity). For this section, we will mostly follow Carroll[1], who does it very simply and elegantly. Wald[4] also has a nice treatment. In Minkowski space, a constant-acceleration trajectory is a hyperbola:

$$x(\tau) = (1/a)cosh(a\tau) \tag{16}$$

$$t(\tau) = (1/a)sinh(a\tau) \tag{17}$$

We can choose a new set of coordinates which are adapted to this motion:

$$x = (1/a)e^{a\xi}cosh(a\eta) \tag{18}$$

$$t = (1/a)e^{a\xi}sinh(a\eta) \tag{19}$$

The metric will now take the following form:

$$ds^2 = -dt^2 + dx^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) \tag{20}$$

(We are only working in 1+1 dimension, as the other dimensions don't play an important role in the calculation). Minkowski space in these coordinates is referred to as "Rindler Space", although of course it's just minkowski space in a different coordinatization. Note that these coordinates only cover the region $x > |t|$. For the other regions, we need to analytically extend these coordinates. For the purposes of our calculation, we'll only need to extend this to cover the region $|x| > |t|$. To do this, we simply note that our timelike coordinate, η is associated with the following timelike killing vector:

$$\frac{\partial}{\partial\eta} = \frac{\partial t}{\partial\eta}\frac{\partial}{\partial t} + \frac{\partial x}{\partial\eta}\frac{\partial}{\partial x} \tag{21}$$

$$= e^{a\xi} cosh(a\eta)\frac{\partial}{\partial t} + e^{a\xi} sinh(a\eta)\frac{\partial}{\partial x} \tag{22}$$

$$= ax\frac{\partial}{\partial t} + at\frac{\partial}{\partial x} \tag{23}$$

This vector naturally extends to region (II), where it clearly runs opposite to the flow of time. For this reason, the analytic extension of these coordinates to region II will look like the following:

$$x = -(1/a)e^{a\xi} cosh(a\eta) \tag{24}$$

$$t = -(1/a)e^{a\xi} sinh(a\eta) \tag{25}$$

Now that we have our coordinates, let us find positive-frequency solutions to the klein-gordon equation in these coordinates. For the sake of simplicity, I assume a massless scalar field:

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu\phi) = 0 \tag{26}$$

Notice now that the conformal factor in my metric exactly cancels with the volume element, so that the equation looks identical to its counterpart in minkowski coorinates:

$$(\partial_\xi^2 - \partial_\eta^2)\phi = 0 \tag{27}$$

We can just write down solutions to this equation:

$$g(k) = e^{-i\omega\eta + ik\xi} \tag{28}$$

where $\omega = |k|$. Of course, this is only a positive-frequency solution in region I. In region II, the complex conjugate is a positive frequency solution. It simplifies things if we write our solutions as only having support in either region I or region II. In other words,

$$\begin{array}{rcl} g^{(1)}(k) = & e^{-i\omega\eta+ik\xi} & (I) \\ = & 0 & (II) \\ g^{(2)}(k) = & 0 & (I) \\ = & e^{i\omega\eta+ik\xi} & (II) \end{array} \tag{29}$$

Now we can express our field in terms of these modes:

$$\phi = \int d^3k\{b_k^{(1)} g^{(1)}(k) + b_k^{(2)} g^{(2)}(k) + b_k^{(1)\dagger} g^{(1)*}(k) + b_k^{(2)\dagger} g^{(2)*}(k)\} \tag{30}$$

In order to calculate the particle content seen by the Rindler observer in Minkowski space, we would like to express these modes in terms of Minkowski plane waves. In fact, this would be an extremely tedious calculation, and Unruh found a shortcut. In fact, we don't need to find a transformation from the $g(k)$ to pure Minkowski modes. All we need to do is find a transformation to some set of functions which are purely positive frequency modes in Minkowski space. In other words, we express our field in the following way:

$$\phi = \int d^3k\{c_k h(k) + c_k^\dagger h^*(k)\} \tag{31}$$

Where $h(k)$ is made out of purely positive-frequency modes in Minkowski space. This way, we are guaranteed that c_k will annihilate the Minkowski vacuum, and doing the Bogolubov transformation from the b's to the c's will tell us the particle content in Rindler space. Now, since our criterion on the $h(k)$'s has been loosened, I can simply find $h(k)$ by finding a positive frequency expression which is continuous and well-defined on the timeslice $t = 0$ and analytically continuing throughout all spacetime. This turns out to be quite simple, if I note the following is true in region (I):

$$a(x-t) = e^{a\xi - a\eta} \tag{32}$$
$$(a(x-t))^{i\omega/a} = e^{-i\omega\eta + i\omega\xi} = g^{(1)}(k) \tag{33}$$

(I am choosing positive wavevector solutions, but the results I'll acquire in a moment are identical for negative wavevectors). Now, in region (II) I have

$$-a(x-t) = e^{a\xi - a\eta} \tag{34}$$
$$(-a(x-t))^{i\omega/a} = e^{-i\omega\eta + i\omega\xi} = g^{(2)*}(-k) \tag{35}$$
$$(e^{-i\pi}a(x-t))^{i\omega/a} = g^{(2)*}(-k) \tag{36}$$
$$e^{\pi\omega/a}(a(x-t))^{i\omega/a} = g^{(2)*}(-k) \tag{37}$$
$$(a(x-t))^{i\omega/a} = e^{-\pi\omega/a}g^{(2)*}(-k) \tag{38}$$

The left hand side now has the same form in both regions, and we have our h(k):

$$h(k) = (g^{(1)}(k) + e^{-\pi\omega/a}g^{(2)*}(-k))N \tag{39}$$

Where N is some normalization. The Bogolubov transformation is then

$$b_k = (c_k + e^{-\pi\omega/a}c_k^\dagger)N \tag{40}$$

and it is easy to see that the normalization we need is

$$b_k = \frac{e^{\pi\omega/2a}c_k + e^{-\pi\omega/2a}c_k^\dagger}{\sqrt{2sinh(\pi\omega/a)}} \tag{41}$$

(Note that I've dropped my superscripts (1) and (2) labelling my modes. Assuredly, there are $c_k^{(1)}$'s and $c_k^{(2)}$'s, but this notation merely distracts us from the main event, which is the particle content seen by the normal observer in Rindler space measuring the Minkowski vacuum:

$$\langle N_R(k)\rangle = \langle 0_M|b_k^\dagger b_k|0_M\rangle = \frac{e^{-\pi\omega/a}}{2sinh(\pi\omega/a)} = \frac{1}{e^{2\pi\omega/a}-1} \tag{42}$$

It's a thermal spectrum, with temperature

$$T = \frac{a}{2\pi} \tag{43}$$

At this point, it may instructive to add back in some of the dimensionful quantities which we typically set to unity:

$$kT = \frac{a}{2\pi}\frac{\hbar}{c} \tag{44}$$

What this tells us is that in order to detect a spectrum of particles at a typical energy scale, we need to be accelerating fast enough that we can reach relativistic speeds in the course of the lifetime of the particles given by the energy-time uncertainty relation. In fact, this can help us to gain another interpretation of this effect. Notice that this mismatch of positive and negative frequency solutions seemed to stem from the fact that Rindler space is split up into these two seemingly incompatible regions. In fact, this is the very heart of the mechanism, because we note that Rindler space has a horizon at $x = t$. Nothing that happens to the left of this region will ever affect the accelerating observer, who is confined to region (I). From the observer's perspective, a particle-antiparticle pair can be produced at this horizon without violating energy conservation, as long as one particle is produced on either side of the horizon (This is the same interpretation that is often given for Hawking radiation, though in the latter case the event horizon is observer-independent).

3 The Role of Time

Actually, there is a much easier way to calculate the temperature measured by an accelerating observer, which is instructive in its own way[1]. We can simply evaluate the two-point function for an accelerating observer, and compare it with the two-point function of a stationary observer in a thermal background. First, we calculate the vacuum state two-point function for a stationary observer, which will only be a function of the invariant interval, $\tau = \sqrt{(t2-t1)^2 - (x2-x1)^2}$:

$$D_0(t) = \langle 0|\phi(t,0)\phi(0,0)|0\rangle \tag{45}$$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}e^{-ikt} = -\frac{1}{4\pi t^2} \tag{46}$$

The same calculation at finite temperature is straightforward:

$$D_T(t) = \langle T|\phi(t,0)\phi(0,0)|T\rangle \tag{47}$$

$$= D_0(t) - \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}\langle T|a_k^\dagger a_k|T\rangle(e^{-ikt} + e^{ikt}) \tag{48}$$

$$= D_0(t) - \int \frac{d^3k}{(2\pi)^3}\frac{1}{2k}(\frac{1}{e^{\beta k}-1})(e^{-ikt} + e^{ikt}) \tag{49}$$

(50)

[1]This section was inspired by a series of lectures by Gruzinov[3]

We can express the term in the integrand as a geometric series:

$$D_T(t) = D_0(t) + \sum_{m=1}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} e^{-m\beta k}(e^{-ikt} + e^{ikt}) \tag{51}$$

$$D_T(t) = D_0(t) + \sum_{m=1}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} (e^{-ikt-km\beta} + e^{ikt-km\beta}) \tag{52}$$

$$D_T(t) = \sum_{m=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{2k} (e^{-ik(t-im\beta)}) \tag{53}$$

$$D_T(t) = \sum_{m=-\infty}^{\infty} D_0(t - im\beta) \tag{54}$$

The two-point function measured in a thermal background is essentially identical to that of the vacuum state, except that it is periodic in imaginary time with period $i\beta$. Now let us look at the two-point function as measured by an accelerated observer. For this, we remember that the observer is on a hyperbolic trajectory, so

$$t_0 = t(0) = 0 \tag{55}$$

$$x_0 = x(0) = 1/a \tag{56}$$

$$t_1 = t(\tau) = (1/a)sinh(a\tau) \tag{57}$$

$$x_1 = x(\tau) = (1/a)cosh(a\tau) \tag{58}$$

$$\Delta\tau^2 = (t_1 - t_0)^2 - (x_1 - x_0)^2 \tag{59}$$

$$= (1/a)^2(sinh^2(a\tau) - (cosh(a\tau) - 1)^2) \tag{60}$$

To find the two-point function measured by the accelerated observer, we simply evaluate the vacuum two-point function on this invariant interval:

$$D_A(\tau) = D_0(\Delta\tau) \tag{61}$$

$$= \frac{a^2}{4\pi(sinh^2(a\tau) - (cosh(a\tau) - 1)^2)} \tag{62}$$

$$= \frac{a^2}{4\pi} \frac{1}{sinh^2(a\tau) - cosh^2(a\tau) + 2cosh(a\tau) - 1} \tag{63}$$

$$= \frac{a^2}{4\pi} \frac{1}{2cosh(a\tau) - 2} \tag{64}$$

$$= \frac{a^2}{4\pi} \frac{1}{4sinh^2(a\tau/2)} \tag{65}$$

This is a meromorphic function, and hence it is entirely determined by the locations and residues of its poles. The poles exactly correspond to the zeroes of the sinh function,

$$sinh(m\pi i) = 0, \forall m \in \mathbb{Z} \tag{66}$$

Now I can express this function as a sum over the poles:

$$D_A(\tau) = \sum_{m=-\infty}^{\infty} \frac{a^2}{4\pi} \frac{1}{4(a\tau/2 - im\pi)^2} \tag{67}$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{4\pi(\tau - 2\pi im/a)^2} \tag{68}$$

$$= \sum_{m=-\infty}^{\infty} D_0(\tau - 2\pi im/a) \tag{69}$$

The accelerated observer measures the same two-point function as a stationary observer in a thermal background. The function is periodic in imaginary time, and by equating the periodicities we get

$$i\beta = 2\pi i/a \tag{70}$$

$$T = \frac{a}{2\pi} \tag{71}$$

This periodicity in imaginary time can be traced back to a symmetry of Rindler space. Let us look at Rindler space (20) again, but change coordinates so that the spatial metric is flat:

$$r = (1/a)e^{a\xi} \tag{72}$$

$$dr = e^{a\xi}d\xi \tag{73}$$

$$ds^2 = e^{2a\xi}(-d\eta^2 + d\xi^2) = -a^2r^2d\eta^2 + dr^2 \tag{74}$$

To see that this metric gives me a periodic structure in imaginary time, I perform a Wick rotation $t \to i\tau$:

$$ds^2 = dr^2 + r^2d(a\tau)^2 \tag{75}$$

This just looks like minkowski space in polar coordinates, and if there is no real singularity at r = 0 this will be periodic in my angular coordinate:

$$a\tau \to a\tau + 2\pi \tag{76}$$

Indeed, Rindler space is periodic in imaginary time, with period $2\pi/a$. Given this observation, we now have a much faster method for calculating the temperature induced by general relativistic quantum effects. For example, we can now very easily calculate the temperature of Hawking radiation. The schwarzchild metric is

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{dx^2}{1 - \frac{2M}{r}} + r^2d\Omega^2 \tag{77}$$

We ignore the angular part of this metric and explore points close to the event horizon, $r = 2M + x$, for small x:

$$ds^2 = -(1 - \frac{2M}{2M + x})dt^2 + \frac{dx^2}{1 - \frac{2M}{2M+x}} \tag{78}$$

$$ds^2 = -(x/2M)dt^2 + \frac{dx^2}{x/2M} \tag{79}$$

Now we perform another coordinate transformation:

$$\rho = 2\sqrt{2Mx} \tag{80}$$

$$x = \rho^2/8M \tag{81}$$

$$d\rho = dx/\sqrt{x/2M} \tag{82}$$

$$ds^2 = -\frac{\rho^2}{16M^2}dt^2 + d\rho^2 \tag{83}$$

Now it's starting to look familiar. Again, we perform a Wick rotation and get

$$ds^2 = \rho^2 d(\frac{\tau}{4M})^2 + d\rho^2 \tag{84}$$

So, this is periodic in imaginary time with periodicity

$$\frac{\tau}{4M} \to \frac{\tau}{4M} + 2\pi \tag{85}$$

$$\tau \to \tau + 8\pi M \tag{86}$$

And so, the temperature of Hawking radiation is

$$T = \frac{1}{8\pi M} \tag{87}$$

It is very interesting that we are able to use this same method to calculate two fundamentally different quantities; the Unruh effect is observer-dependent, Hawking radiation is not. As always, using clever tricks like this can be a double-edged sword. As we have seen from our example of Hawking radiation, it is possible to get an exact answer without really asking the question. We should strive as theorists for a better understanding of the complex structure of time and how it relates to effects like this.

4 Particle Creation by a Time-Dependent Gravitational Field

Aside from these interesting theoretical implications, we can also use this theoretical framework to calculate some important physics, for example how a time-dependent gravitational field can create particles[2]. This would, for example, be important in understanding some of the output of inflation. The idea is very similar to what we've discussed. Particles are being produced because of the mixing between positive and negative frequency states when the metric becomes dynamic. The simplest way to understand this is to first assume

[2]Ford[2] gives a much better treatment of this subject. I mostly follow his lead in this section but my calculations involve immense simplification

that the metric is static in the asymptotic past. Then our field operator can be written in terms of the solutions to equation (5) in this region:

$$\phi = \int d^3k\{a_k f(\vec{k}) + a_k^\dagger f^*(\vec{k})\} \tag{88}$$

$f(\vec{k})$ is a positive frequency solution to equation (5), but as soon as time-dependence kicks in, the solutions to this equation will no longer be purely positive frequency in this coordinate system. The a_k annihilate a vacuum state, which we call the "in" vacuum:

$$a_k|0\rangle_{in} = 0 \tag{89}$$

but once the metric starts evolving this will not be the vacuum state. If space-time is also asymptotically flat in the future, then we can define out solutions, out states, and out operators:

$$\phi = \int d^3k\{b_k g(\vec{k}) + b_k^\dagger g^*(\vec{k})\} \tag{90}$$

$$b_k|0\rangle_{out} = 0 \tag{91}$$

And, as before, we can relate them by a Bogolubov transformation:

$$b_k = \alpha_{kk'}a_k + \beta_{kk'}a_k^\dagger \tag{92}$$

and the number of particles created is

$$\langle N_{out}(\vec{k})\rangle =_{in} \langle 0|b^\dagger(\vec{k})b(\vec{k})|0\rangle_{in} = \int d^3k'|\beta_{kk'}|^2 \tag{93}$$

To make things concrete, we consider the specific example of an expanding spatially flat FRW universe.

$$ds^2 = -dt^2 + a^2(t)d\vec{x}^2 = a^2(\eta)(-d\eta^2 + d\vec{x}^2) \tag{94}$$

The calculation can be done in either the comoving time t or conformal time η, but the calculation in η turns out to be much simpler. We look at equation (5) once more, replacing the mass and curvature terms that we previously omitted:

$$-\partial_\eta(a^2(\eta)\partial_\eta\phi) + a^2(\eta)\nabla^2\phi - a^4(\eta)m^2\phi - a^4(\eta)\xi R\phi = 0 \tag{95}$$

Now, to make the first term more manageable, we substitute $\phi = \chi/a(\eta)$:

$$-\partial_\eta(a^2(\eta)\partial_\eta\phi) = -\partial_\eta(a(\eta)\partial_\eta\chi) - \partial_\eta(\chi\partial_\eta a(\eta)) \tag{96}$$

$$= -\partial_\eta a(\eta)\partial_\eta\chi - a(\eta)\partial_\eta^2\chi + \partial_\eta\chi\partial_\eta a(\eta) + \chi\partial_\eta^2 a(\eta) \tag{97}$$

Two of these terms cancel, and we get:

$$-\partial_\eta(a^2(\eta)\partial_\eta\phi) = \chi\partial_\eta^2 a(\eta) - a(\eta)\partial_\eta^2\chi \tag{98}$$

After subtituting this term, our equation of motion for χ is now

$$(-\partial_\eta^2 + \nabla^2)\chi - a^2(\eta)(m^2 + \xi R - a^{-3}(\eta)\partial_\eta^2 a(\eta))\chi = 0 \tag{99}$$

But we recall that for a spatially flat expanding FRW universe, the scalar curvature is just given by

$$R = \frac{6}{a^3(\eta)}\partial_\eta^2 a(\eta) \tag{100}$$

which is one of our terms. Collecting everything together,

$$(-\partial_\eta^2 + \nabla^2)\chi - a^2(\eta)(m^2 + (\xi - \frac{1}{6})R)\chi = 0 \tag{101}$$

This is the reason many theorists introduce the scalar curvature term in the first place. By choosing $\chi = 1/6$, we would make a massless theory conformally invariant. Of course, if it's conformally invariant, we would get no particle production, as we could perform a conformal transformation to flat space. So, we'll keep the mass term but set $\chi = 1/6$. Our equation of motion is now

$$(-\partial_\eta^2 + \nabla^2)\chi - a^2(\eta)m^2\chi = 0 \tag{102}$$

This will give us a dispersion relation if we expand in plane waves $\chi = e^{-i\omega(\eta)\eta + i\vec{k}\cdot x}$, to get an equation for $\omega(\eta)$:

$$i\dot{\omega} + \omega^2 = \vec{k}^2 + a^2(\eta)m^2 \tag{103}$$

As we can see, the time-dependent metric is altering our dispersion relation. This should not be too surprising, since typically we write dispersion relations in the covariant form

$$k^\mu k_\mu + m^2 = 0 \tag{104}$$

$$\frac{1}{a^2(\eta)}(-\omega^2 + \vec{k}^2) + m^2 = 0 \tag{105}$$

Which, if we ignore the time derivative term, reduces to (103). Now, instead of doing something realistic, let's entertain a toy problem of a universe that explodes then contracts in an infinitesimal amount of time. Of course, this could perhaps be approximated by some mildly realistic finite local perturbation, but our purposes are purely pedagogical. The scale factor I use is

$$a(\eta) = 1 + i\tau\delta(\eta) \tag{106}$$

Though this "explosion" is instantaneous, we can consider τ to be a characteristic timescale. Now our equation of motion is

$$-\partial_\eta^2\chi = (k^2 + m^2(1 + i\tau\delta(\eta))\chi \tag{107}$$

The solutions to this for all $t \neq 0$ are merely plane waves with the usual dispersion relation. To find the relationship between the in states and the out states, we integrate over an infinitesimal region about $t = 0$:

$$-\partial_\eta \chi|_{-\epsilon}^{\epsilon} = im^2\tau\chi \tag{108}$$

Now we use our plane wave solutions to take the other derivative:

$$i(\omega_+ - \omega -)\chi = im^2\tau\chi \tag{109}$$

So, we see there is a discontinuity in the frequency:

$$\Delta\omega = m^2\tau \tag{110}$$

For most states, this will send a pure plane wave into a collection of arbitrary plane waves, because the new frequency it gets scattered to won't generically satisfy the dispersion relation. However, for every value of τ, there will be one positive frequency state that gets sent exactly into its negative-frequency counterpart, or vice-versa. The resonant frequency is

$$\omega_* = \frac{1}{2}\Delta\omega = \frac{1}{2}m^2\tau \tag{111}$$

It is not too insane an extrapolation to expect that if we vary the metric on timescales of order τ, we can expect to excite states efficiently at this energy scale.

References

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